

# PICONE'S IDENTITY FOR $p$ -BIHARMONIC OPERATOR AND ITS APPLICATIONS

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**ABSTRACT.** In this article we prove the nonlinear analogue of Picone's identity for  $p$ -biharmonic operator. As an application of our result we show that the Morse index of the zero solution to a  $p$ -biharmonic boundary value problem is 0. We also prove a Hardy type inequality and Sturmian comparison principle. We also show the strict monotonicity of the principle eigenvalue and linear relationship between the solutions of a system of singular  $p$ -biharmonic system.

## 1. INTRODUCTION

The classical Picone's identity says that for differentiable functions  $v > 0$  and  $u \geq 0$ ,

$$(1.1) \quad |\nabla u|^2 + \frac{u^2}{v^2} |\nabla v|^2 - 2 \frac{u}{v} \nabla u \nabla v = |\nabla u|^2 - \nabla \left( \frac{u^2}{v} \right) \nabla v \geq 0.$$

(1.1) has an enormous applications to second-order elliptic equations and systems, see for instance, [1, 2, 3, 12] and the references therein. Nonlinear analogue of (1.1) is established by J. Tyagi [13]. In order to apply (1.1) to  $p$ -Laplace equations, (1.1) is extended by W. Allegretto and Y.X.Huang [4]. Nonlinear analogue of Picone's type identity for  $p$ -Laplace equations is established by K. Bal [5].

In [6], D.R.Dunninger established a Picone identity for a class of fourth order elliptic differential inequalities. This identity says that if  $u, v, a\Delta u, A\Delta v$  are twice continuously differentiable functions with  $v(x) \neq 0$  and  $a$  and  $A$  are positive weights, then

$$(1.2) \quad \begin{aligned} & \operatorname{div} \left[ u \nabla (a \Delta u) - a \Delta u \nabla u - \frac{u^2}{v} \nabla (A \Delta v) + A \Delta v \nabla \left( \frac{u^2}{v} \right) \right] \\ &= - \frac{u^2}{v} \Delta (A \Delta v) + u \Delta (a \Delta u) + (A - a) (\Delta u)^2 \\ &- A \left( \Delta u - \frac{u}{v} \Delta v \right)^2 + A \frac{2 \Delta v}{v} \left( \nabla u - \frac{u}{v} \nabla v \right)^2. \end{aligned}$$

With some simplifications in (1.2), we obtain the following identity:

Let  $u$  and  $v$  be twice continuously differentiable functions in  $\Omega$  such that  $v > 0$ ,  $-\Delta v > 0$  in  $\Omega$ . Denote

$$(1.3) \quad L(u, v) = \left( \Delta u - \frac{u}{v} \Delta v \right)^2 - \frac{2 \Delta v}{v} \left( \nabla u - \frac{u}{v} \nabla v \right)^2.$$

2010 *Mathematics Subject Classification.* Primary 35J91; Secondary 35B60.

*Key words and phrases.*  $p$ -biharmonic, Picone's identity, Morse index.

Submitted 19-03-2015. Published—.

$$R(u, v) = |\Delta u|^2 - \Delta \left( \frac{u^2}{v} \right) \Delta v.$$

Then (i)  $L(u, v) = R(u, v)$  (ii)  $L(u, v) \geq 0$  and (iii)  $L(u, v) = 0$  in  $\Omega$  if and only if  $u = \alpha v$  for some  $\alpha \in \mathbb{R}$ .

Nonlinear analogue of (1.3) is established by G. Dwivedi and J. Tyagi [7]. Picone's identity for p-biharmonic operator is established by J. Jaroš [10]. Picone's identity for elliptic differential operators are discussed by N. Yoshida [14] and J. Jaroš [11].

In this article we establish the nonlinear analogue of Picone's identity for p-biharmonic operator. We also discuss some qualitative results in the spirit of W. Allegretto and Y.X.Huang [4] and J. Tyagi [13].

The plan of the paper is as follows: Section 2 deals with nonlinear analogue of Picone's identity. In section 3, we give several application of Picone's identity to p-biharmonic equations.

## 2. MAIN RESULTS

Throughout this article, we assume the following hypotheses, unless otherwise stated.

- (i)  $\Delta_p^2 := \Delta(|\Delta u|^{p-2} \Delta u)$ , denotes  $p$ -biharmonic operator.
- (ii)  $\Omega$  denotes any domain in  $\mathbb{R}^n$ .
- (iii)  $1 < p < \infty$ .
- (iv)  $f : \mathbb{R} \rightarrow (0, \infty)$  be a  $C^2$  function.

First we state Young's inequality, which will be used later.

**Lemma 2.1.** *If  $a$  and  $b$  are two nonnegative real numbers and  $p$  and  $q$  are such that  $\frac{1}{p} + \frac{1}{q} = 1$ , then*

$$(2.1) \quad ab \leq \frac{a^p}{p} + \frac{b^q}{q},$$

*equality holds if and only if  $a^p = b^q$ .*

*Proof.* For proof we refer to [8]. □

Next we give Picone's identity for  $p$ -biharmonic operator.

**Lemma 2.2.** *Let  $u \geq 0$ ,  $v > 0$  be twice continuously differentiable functions in  $\Omega$  and  $-\Delta v > 0$  in  $\Omega$ . Denote*

$$\begin{aligned} R(u, v) &= |\Delta u|^p - \Delta \left( \frac{u^p}{v^{p-1}} \right) |\Delta v|^{p-2} \Delta v \\ L(u, v) &= |\Delta u|^p + \frac{(p-1)u^p}{v^p} |\Delta v|^p - \frac{pu^{p-1}}{v^{p-1}} |\Delta v|^{p-2} \Delta u \Delta v \\ &\quad - \frac{p(p-1)u^{p-2}}{v^{p-1}} \Delta v |\Delta v|^{p-2} \left( \nabla u - \frac{u}{v} \nabla v \right)^2. \end{aligned}$$

Then (i)  $L(u, v) = R(u, v)$ , (ii)  $L(u, v) \geq 0$ , (iii)  $L(u, v) = 0$  in  $\Omega$  if and only if  $u = \alpha v$  for some  $\alpha \in \mathbb{R}$ .

*Proof.* Let us expand  $R(u, v)$ :

$$R(u, v) = |\Delta u|^p - \Delta \left( \frac{u^p}{v^{p-1}} \right) |\Delta v|^{p-2} \Delta v$$

$$\begin{aligned}
&= |\Delta u|^p + \frac{(p-1)u^p}{v^p} |\Delta v|^p - \frac{pu^{p-1}}{v^{p-1}} |\Delta v|^{p-2} \Delta u \Delta v \\
&\quad + \frac{2p(p-1)u^{p-1}}{v^p} |\Delta v|^{p-2} (\nabla u \cdot \nabla v) \Delta v - \frac{p(p-1)u^{p-2}}{v^{p-1}} |\nabla u|^2 |\Delta v|^{p-2} \Delta v \\
&\quad - \frac{p(p-1)u^p}{v^{p+1}} |\nabla v|^2 |\Delta v|^{p-2} \Delta v \\
&= \underbrace{\left( |\Delta u|^p + \frac{(p-1)u^p}{v^p} |\Delta v|^p - \frac{pu^{p-1}}{v^{p-1}} |\Delta v|^p |\Delta u| |\Delta v| \right)}_{(I)} \\
&\quad + \underbrace{\frac{pu^{p-1}}{v^{p-1}} |\Delta v|^{p-2} (|\Delta u| |\Delta v| - \Delta u \Delta v)}_{(II)} \\
&\quad + \underbrace{\frac{2p(p-1)u^{p-1}}{v^p} |\Delta v|^{p-2} (\nabla u \cdot \nabla v) \Delta v - \frac{p(p-1)u^{p-2}}{v^{p-1}} |\nabla u|^2 |\Delta v|^{p-2} \Delta v}_{(III)} \\
&\quad - \underbrace{\frac{p(p-1)u^p}{v^{p+1}} |\nabla v|^2 |\Delta v|^{p-2} \Delta v}_{(IV)}.
\end{aligned}$$

First consider (III):

$$\begin{aligned}
(III) &= \frac{p(p-1)u^{p-2}}{v^{p-1}} \Delta v |\Delta v|^{p-2} \left( \frac{2u}{v} \nabla u \cdot \nabla v - |\nabla u|^2 - \frac{u^2}{v^2} |\nabla|^2 \right) \\
&= -\frac{p(p-1)u^{p-2}}{v^{p-1}} \Delta v |\Delta v|^{p-2} \left( \nabla u - \frac{u}{v} \nabla v \right)^2.
\end{aligned}$$

This shows that  $R(u, v) = L(u, v)$ .

Since  $u \geq 0$ ,  $v > 0$ ,  $-\Delta v > 0$ , we get (III)  $\geq 0$ .

Next we consider (II). Since  $|\Delta u| |\Delta v| \geq \Delta u \Delta v$ , therefore, (II)  $\geq 0$ .

Now consider (I):

$$(I) = |\Delta u|^p + \frac{(p-1)u^p}{v^p} |\Delta v|^p - \frac{pu^{p-1}}{v^{p-1}} |\Delta v|^p |\Delta u| |\Delta v|.$$

By Lemma 2.1, with  $a = |\Delta u|$  and  $b = \frac{|u^{p-1}| |\Delta v|^{p-1}}{v^{p-1}}$ , we get

$$\frac{pu^{p-1}}{v^{p-1}} |\Delta u| |\Delta v|^{p-1} \leq |\Delta u|^p + \frac{p}{q} \frac{u^p}{v^p} |\Delta v|^p,$$

which gives

$$|\Delta u|^p + \frac{(p-1)u^p}{v^p} |\Delta v|^p - \frac{pu^{p-1}}{v^{p-1}} |\Delta v|^p |\Delta u| |\Delta v| \geq 0.$$

This proves that (I)  $\geq 0$ . This proves the (ii), that is,  $L(u, v) \geq 0$ .

Now  $L(u, v) = 0$  in  $\Omega$  implies that  $\nabla u - \frac{u}{v} \nabla v = 0$ , provided  $u(x_0) \neq 0$  for some  $x_0 \in \Omega$ . This gives  $\nabla \left( \frac{u}{v} \right) = 0$ , that is,  $u = \alpha v$  for some  $\alpha \in \mathbb{R}$ . This completes the proof.  $\square$

Now we establish the nonlinear analogue of Picone's identity for  $p$ -biharmonic operator.

**Lemma 2.3.** *Let  $u$  and  $v$  be twice continuously differentiable functions in  $\Omega$  such that  $u \geq 0$  and  $-\Delta v > 0$  in  $\Omega$ . Let  $f: \mathbb{R} \rightarrow (0, \infty)$  be a  $C^2$  function such that  $f'(y) \geq (p-1)[f(y)]^{\frac{p-1}{p-2}}$  and  $f''(y) \leq 0, \forall y \in \mathbb{R}$ . Denote*

$$\begin{aligned} L(u, v) &= |\Delta u|^p - \frac{pu^{p-1}|\Delta v|^{p-2}\Delta u\Delta v}{f} + \frac{u^p f'(v)|\Delta v|^p}{f^2} \\ &\quad + \frac{u^p f''(v)|\nabla v|^2|\Delta v|^{p-2}\Delta v}{f^2} \\ &\quad - \frac{1}{2} \frac{\Delta v|\Delta v|^{p-2}u^{p-2}}{f} \left[ \left( \frac{2uf'}{f} \nabla v - p\nabla u \right)^2 + p(p-1)|\nabla u|^2 \right]. \\ R(u, v) &= |\Delta u|^p - \Delta \left( \frac{u^p}{f(v)} \right) |\Delta v|^{p-2}\Delta v. \end{aligned}$$

Then (i)  $L(u, v) = R(u, v)$ , (ii)  $L(u, v) \geq 0$ , (iii)  $L(u, v) = 0$  in  $\Omega$  if and only if  $u = \alpha v$  for some  $\alpha \in \mathbb{R}$ .

*Proof.* Let us expand the  $R(u, v)$  :

$$\begin{aligned} R(u, v) &= |\Delta u|^p - \Delta \left( \frac{u^p}{f(v)} \right) |\Delta v|^{p-2}\Delta v \\ &= |\Delta u|^p + \frac{u^p f'(v)|\Delta v|^p}{f^2} - \frac{pu^{p-1}|\Delta v|^{p-2}\Delta u\Delta v}{f(v)} \\ &\quad - \frac{p(p-1)u^{p-2}|\nabla u|^2|\Delta v|^{p-2}\Delta v}{f} - \frac{2u^p f'^2|\nabla v|^2|\Delta v|^{p-2}\Delta v}{f^3} \\ &\quad + \frac{2pf'(v)u^{p-1}(\nabla u \cdot \nabla v)|\Delta v|^{p-2}\Delta v}{f^2} + \frac{u^p f''(v)|\nabla v|^2|\Delta v|^{p-2}\Delta v}{f^2} \\ &= \underbrace{\left( |\Delta u|^p + \frac{u^p f'(v)|\Delta v|^p}{f^2} - \frac{pu^{p-1}|\Delta v|^{p-2}\Delta u\Delta v}{f(v)} \right)}_{(I)} \\ &\quad - \underbrace{\frac{\Delta v|\Delta v|^{p-2}u^{p-2}}{f} \left( p(p-1)|\nabla u|^2 + 2\frac{u^2 f'^2}{f^2}|\nabla v|^2 - \frac{2pf'u}{f}(\nabla u \cdot \nabla v) \right)}_{(II)} \\ &\quad + \underbrace{\frac{u^p f''(v)|\nabla v|^2|\Delta v|^{p-2}\Delta v}{f^2}}_{(III)}. \end{aligned}$$

First we consider (II).

$$\begin{aligned} (II) &= -\frac{\Delta v|\Delta v|^{p-2}u^{p-2}}{f} \left[ p(p-1)|\nabla u|^2 + \frac{2u^2 f'^2}{f^2}|\nabla v|^2 - \frac{2pf'u}{f}(\nabla u \cdot \nabla v) \right] \\ &\quad + \left( \frac{p\nabla u}{\sqrt{2}} \right)^2 - \left( \frac{p\nabla u}{\sqrt{2}} \right)^2 \\ &= -\frac{1}{2} \frac{\Delta v|\Delta v|^{p-2}u^{p-2}}{f} \left[ \left( \frac{2uf'}{f} \nabla v - p\nabla u \right)^2 + p(p-1)|\nabla u|^2 \right]. \end{aligned}$$

This completes (i), that is,  $L(u, v) = R(u, v)$ . Also  $(II) \geq 0$ , since  $-\Delta v > 0$ . Next consider (I).

$$(I) = \left( |\Delta u|^p + \frac{u^p f'(v) |\Delta v|^p}{f^2} - \frac{pu^{p-1} |\Delta v|^{p-2} |\Delta u| |\Delta v|}{f(v)} \right) + \frac{pu^{p-1} |\Delta v|^{p-2}}{f} (|\Delta u| |\Delta v| - \Delta u \Delta v),$$

clearly second term of above equation is nonnegative. So on using Young's inequality (Lemma 2.1) with  $a = |\Delta u|$ ,  $b = \frac{(u |\Delta v|)^{p-1}}{f}$ , we get

$$\frac{|\Delta u| u^{p-1} |\Delta v|^{p-1}}{f} \leq \frac{|\Delta u|^p}{p} + \frac{(u |\Delta v|)^{(p-1)q}}{q f^q}$$

$$p \frac{|\Delta u| u^{p-1} |\Delta v|^{p-1}}{f} \leq |\Delta u|^p + (p-1) \frac{(u |\Delta v|)^{(p-1)q}}{f^q},$$

equality holds when

$$(2.2) \quad |\Delta u| = \frac{u |\Delta v|}{[f(v)]^{\frac{q}{p}}}.$$

Now on using  $f'(y) \geq (p-1)[f(y)]^{\frac{p-2}{p-1}}$ , we get

$$|\Delta u|^p + \frac{(u^p f'(v) |\Delta v|^p)}{f^2} - \frac{|\Delta u| u^{p-1} |\Delta v|^{p-1}}{f} \geq 0$$

and equality holds when

$$(2.3) \quad f'(y) = (p-1)[f(y)]^{\frac{p-2}{p-1}}$$

This gives  $(I) \geq 0$ .

Now  $(III) \geq 0$ , since  $-\Delta v > 0$  and  $f''(v) \leq 0$ . This proves (ii).  $\square$

### 3. APPLICATIONS

In this section we will give some applications of nonlinear Picone's identity following the spirit of [4].

**Hardy type result.** We start with establishing a Hardy type inequality for p-biharmonic operator.

**Theorem 3.1.** *Let there be a  $v \in C_c^\infty(\Omega)$  such that*

$$\Delta(|\Delta v|^{p-2} \Delta v) \geq \lambda g f(v), \quad v > 0 \text{ in } \Omega, \quad -\Delta v > 0 \text{ in } \Omega,$$

*for some  $\lambda > 0$  and a nonnegative continuous function  $g$  then for any  $u \in C_c^\infty(\Omega)$ ;  $u \geq 0$  it holds that*

$$(3.1) \quad \int_{\Omega} |\Delta u|^p dx \geq \lambda \int_{\Omega} g |u|^p dx,$$

*where  $f$  satisfies  $f'(y) \geq (p-1)[f(y)]^{\frac{p-2}{p-1}}$ .*

*Proof.* Take  $\phi \in C_c^\infty(\Omega)$ ,  $\phi > 0$ . By Lemma 2.3, we have

$$0 \leq \int_{\Omega} L(\phi, v) dx$$

$$= \int_{\Omega} R(\phi, v) dx = \int_{\Omega} \left( |\Delta \phi|^p - \Delta \left( \frac{\phi^p}{f(v)} \right) |\Delta v|^{p-2} \Delta v \right) dx$$

$$\begin{aligned}
&= \int_{\Omega} |\Delta \phi|^p dx - \int_{\Omega} \frac{\phi^p}{f(v)} \Delta(|\Delta v|^{p-2} \Delta v) dx \\
&\leq \int_{\Omega} |\Delta \phi|^p dx - \lambda \int_{\Omega} \phi^p g dx
\end{aligned}$$

letting  $\phi \rightarrow u$ , we get

$$\int_{\Omega} |\Delta u|^p dx \geq \lambda \int_{\Omega} g |u|^p dx.$$

□

**Strumium comparison principle.** Comparison principles play vital role in study of partial differential equations. Here, we establish nonlinear version of Strumium comparison principle for p-biharmonic operator.

**Theorem 3.2.** *Let  $f_1$  and  $f_2$  are two weight functions such that  $f_1 < f_2$  and  $f$  satisfies  $f'(y) \geq (p-1)[f(y)]^{\frac{p-2}{p-1}}$ . If there is a positive solution  $u$  satisfying*

$$\begin{aligned}
(3.2) \quad &\Delta_p^2 v = f_1(x) |u|^{p-2} u \text{ in } \Omega \\
&u = 0 = \Delta u \text{ on } \partial\Omega.
\end{aligned}$$

*Then any nontrivial solution  $v$  of*

$$\begin{aligned}
(3.3) \quad &\Delta_p^2 v = f_2(x) f(v) \text{ in } \Omega \\
&u = 0 = \Delta u \text{ on } \partial\Omega,
\end{aligned}$$

*must change sign.*

*Proof.* Let us assume that there exists a solution  $v > 0$  of (3.3) in  $\Omega$ . Then by Picone's identity we have

$$\begin{aligned}
0 &\leq \int_{\Omega} L(u, v) dx = \int_{\Omega} R(u, v) dx \\
&= \int_{\Omega} |\Delta u|^p - \Delta \left( \frac{u^p}{f(v)} \right) |\Delta v|^{p-2} \Delta v dx \\
&= \int_{\Omega} (f_1(x) u^p - f_2(x) u^p) dx \\
&= \int_{\Omega} (f_1 - f_2) u^p dx < 0,
\end{aligned}$$

which is a contradiction. Hence,  $v$  changes sign. □

**Strict Monotonicity of principle eigenvalue in domain.** Consider the indefinite eigenvalue problem

$$\begin{aligned}
(3.4) \quad &\Delta_p^2 u = \lambda g, \text{ in } \Omega, \\
&u = 0 = \Delta u \text{ on } \partial\Omega,
\end{aligned}$$

where  $g(x)$  is indefinite weight function.

**Theorem 3.3.** *Let  $\lambda_1^+(\Omega) > 0$  be the principle eigenvalue of (3.4), then suppose  $\Omega_1 \subset \Omega_2$  and  $\Omega_1 \neq \Omega_2$ . Then  $\lambda_1^+(\Omega_1) > \lambda_1^+(\Omega_2)$ , if both exist.*

*Proof.* Let  $u_i$  be a positive eigenfunction associated with  $\lambda_1^+(\Omega_i)$ ,  $i = 1, 2$ . Evidently for  $\phi \in C_0^\infty(\Omega_1)$ , we have

$$0 \leq \int_{\Omega_1} L(\phi_1, u_2) dx$$

$$\begin{aligned}
&= \int_{\Omega_1} |\Delta \phi|^p dx - \int_{\Omega_1} \frac{\phi^p}{f(u_2)} \Delta (|\Delta u_2|^{p-2} \Delta u_2) dx \\
&= \int_{\Omega_1} |\Delta \phi|^p dx - \lambda_1^+(\Omega_2) \int_{\Omega_1} g(x) \phi^p dx.
\end{aligned}$$

Letting  $\phi \rightarrow u_1$ , we get

$$0 \leq \int_{\Omega_1} L(u_1, u_2) dx = (\lambda_1^+(\Omega_1) - \lambda_1^+(\Omega_2)) \int_{\Omega_1} g \phi^p dx,$$

this gives  $\lambda_1^+(\Omega_1) - \lambda_1^+(\Omega_2) > 0$ , as if  $\lambda_1^+(\Omega_1) = \lambda_1^+(\Omega_2)$ , we conclude that  $u_1 = ku_2$ , which is not possible as  $\Omega_1 \subset \Omega_2$  and  $\Omega_1 \neq \Omega_2$ . This completes the proof.  $\square$

**Quasilinear System with Singular nonlinearity.** We will use Picone's identity to establish a linear relationship between solutions of a quasilinear system with singular nonlinearity. Consider the singular system of elliptic equations

$$\begin{aligned}
(3.5) \quad &\Delta_p^2 u = f(v), \text{ in } \Omega, \\
&\Delta_p^2 v = \frac{(f(v))^2}{u}, \text{ in } \Omega, \\
&u > 0, v > 0 \text{ in } \Omega, \\
&u = 0 = v \text{ on } \partial\Omega, \\
&\Delta u = 0 = \Delta v \text{ on } \partial\Omega,
\end{aligned}$$

where  $f$  satisfies  $f'(y) \geq (p-1)[f(y)]^{\frac{p-2}{p-1}}$ .

**Theorem 3.4.** *Let  $(u, v)$  be a weak solution of (3.5) and  $f$  satisfy  $f'(y) \geq (p-1)[f(y)]^{\frac{p-2}{p-1}}$ , then  $u = c_1 v$ , where  $c_1$  is a constant.*

*Proof.* Let  $(u, v)$  be weak solution of (3.5). Then for any  $\phi_1, \phi_2 \in H^2(\Omega) \cap H_0^1(\Omega)$ , we have

$$(3.6) \quad \int_{\Omega} |\Delta u|^{p-2} \Delta u \Delta \phi_1 dx = \int_{\Omega} f(v) \phi_1 dx$$

$$(3.7) \quad \int_{\Omega} |\Delta v|^{p-2} \Delta v \Delta \phi_2 dx = \int_{\Omega} \frac{f^2(v)}{u^{p-1}} \phi_2 dx.$$

Choosing  $\phi_1 = u$  and  $\phi_2 = \frac{u^p}{f(v)}$  in (3.6) and (3.7) respectively, we get

$$\begin{aligned}
\int_{\Omega} |\Delta u|^p dx &= \int_{\Omega} u f(v) dx \\
&= \int_{\Omega} |\Delta v|^{p-2} \Delta v \Delta \left( \frac{u^2}{f(v)} \right) dx,
\end{aligned}$$

which gives

$$\int_{\Omega} \left( |\Delta u|^p - \Delta v |\Delta v|^{p-2} \Delta \left( \frac{u^2}{f(v)} \right) \right) dx = 0,$$

or

$$\int_{\Omega} R(u, v) dx = 0,$$

this gives  $R(u, v) = 0$ , which in turn implies that  $u = c_1 v$ .  $\square$

**Morse Index.** Let us consider the problem

$$(3.8) \quad \begin{aligned} \Delta_p^2 u &= a(x)f(u), \quad \text{in } \Omega, \\ u &= 0 = \Delta u, \quad \text{on } \partial\Omega. \end{aligned}$$

The morse index of the solution of the (3.8) is the number of negative eigenvalues of the linearized operator

$$\Delta_p^2 - a(x)f'(u)$$

acting on  $H^2(\Omega) \cap H_0^1(\Omega)$ , that is, the number of eigenvalue  $\lambda$  such that  $\lambda < 0$  and the boundary value problem

$$(3.9) \quad \begin{aligned} \Delta_p^2 w - a(x)f'(u)w &= \lambda w, \quad \text{in } \Omega, \\ w &= 0 = \Delta w, \quad \text{on } \partial\Omega \end{aligned}$$

has a nontrivial solution  $w$  in  $H^2(\Omega) \cap H_0^1(\Omega)$ .

**Theorem 3.5.** *Let us consider (3.8). Suppose  $f'(0) \leq 1 \leq f'(s)$ ,  $\forall s \in (0, \infty)$  and  $f(0) = 0$ . Let  $a(x)$  be a positive continuous function in  $\bar{\Omega}$ . Then the trivial solution of (3.8) has morse index 0.*

*Proof.* Let  $v \in H^2(\Omega) \cap H_0^1(\Omega)$  be a positive solution of (3.8). Then

$$(3.10) \quad \int_{\Omega} |\Delta v|^{p-2} \Delta v \Delta \psi \, dx = \int_{\Omega} a(x)f(v)\psi \, dx, \quad \forall \psi \in H^2(\Omega) \cap H_0^1(\Omega).$$

For any  $0 \neq w \in H^2(\Omega) \cap H_0^1(\Omega)$ , let us take  $\psi = \frac{w^2}{f(v)}$  as a test function in (3.10), we get

$$(3.11) \quad \int_{\Omega} |\Delta v|^{p-2} \Delta v \Delta \left( \frac{w^2}{f(v)} \right) dx = \int_{\Omega} a(x)f(v) \frac{w^2}{f(v)} dx,$$

on using  $R(u, v) \geq 0$ , we get

$$(3.12) \quad \int_{\Omega} |\Delta w|^p dx \geq \int_{\Omega} a(x)w^2 dx \geq \int_{\Omega} a(x)f'(0)w^2 dx.$$

Consider the eigenvalue problem associated with the linearization of (3.8) at 0, which is nothing but

$$(3.13) \quad \begin{aligned} \Delta_p^2 w - a(x)f'(0)w &= \lambda w, \quad \text{in } \Omega, \\ w &= 0 = \Delta w, \quad \text{on } \partial\Omega \end{aligned}$$

By variational characterization of the eigenvalue in (3.13), from (3.12), we get that  $\lambda \geq 0$  and corresponding eigenfunction is positive. Which proves the claim.  $\square$

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